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Diagonalising Modular Forms of Half-Integral Weight

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In this paper we decompose the space $S_{k+1/2}(N, \chi)$ (N is a squarefree natural number and χ a quadratic character) with respect to the Hecke operators $T(p^2)(p \nmid N)$ and $C(p^2)(p \mid N)$ operators which arise naturally from the familiar $U(p^2)$ operators and the W-operators. We obtain an analogous result of A. Pizer (*J. Algebra* **83** (1983), 39–64, Theorem 3.10).

1. INTRODUCTION

Let p be a prime number, $k \geqslant 2$, and N be natural numbers with N squarefree. Set

$$M = \begin{cases} N/2 & \text{if } N \text{ is even} \\ N & \text{if } N \text{ is odd.} \end{cases}$$

For a quadratic character χ mod M of conductor t with $\chi(-1) = \varepsilon$, let $\chi_1 = (4\varepsilon/\cdot)\chi$. We denote by $S_{k+1/2}(\Gamma_0(4M), \chi_1)$, the space of cusp forms of weight $k + 1/2$ for $\Gamma_0(4M)$ with character χ_1 and by $S^+_{k+1/2}(\Gamma_0(4M), \chi_1)$ the "+ space" of $S_{k+1/2}(\Gamma_0(4M), \chi_1)$ consisting of forms whose nth Fourier coefficient vanishes whenever $\varepsilon(-1)^k n \equiv 2, 3 \pmod 4$. We further let

$$S_{k+1/2}(N, \chi) = \begin{cases} S_{k+1/2}(\Gamma_0(4M), \chi_1) & \text{if } N \text{ is even} \\ S^+_{k+1/2}(\Gamma_0(4M), \chi_1) & \text{if } N \text{ is odd.} \end{cases}$$

The space of cusp forms of weight $2k$, level N with trivial character is denoted by $S_{2k}(N)$.

In his paper [4], A. Pizer decomposed the space $S_{2k}(N)$ into a direct sum of common eigenspaces with respect to the Hecke operators

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$T(p)(p \nmid N)$ and $C(p)(p \mid N)$ operators which arise naturally from the familiar $U(p)$ operators and the Atkin–Lehner W-operators. He proved that each common eigenspace is of dimension one and contains a normalized form F whose pth Fourier coefficients are eigenvalues of $T(p)(p \nmid N)$ and $C(p)(p \mid N)$.

The aim of this paper is to obtain an analogous decomposition in $S_{k+1/2}(N, \chi)$. We will decompose the space $S_{k+1/2}(N, \chi)$ into a direct sum of common eigenspaces with respect to the Hecke operators $T(p^2)(p \nmid N)$ and the corresponding $C(p^2)(p \mid N)$ operators and prove that each common eigenspace is of dimension one and contains a non-zero form f whose eigenvalues with respect to $T(p^2)(p \nmid N)$ and $C(p^2)(p \mid N)$ are the pth Fourier coefficients of the normalized form $F = f \mid \mathscr{S}_{k,N,\chi} \in S_{2k}(N)$, where $\mathscr{S}_{k,N,\chi}$ is one of the Shimura isomorphism constructed in [3]. We will obtain the result, by proving the commutative rule

$$C(p^2)\,\mathscr{S}_{k,N,\chi} = \mathscr{S}_{k,N,\chi}\,C(p)(p \mid N).$$

2. Notations

For a natural number m, we define the operator $U(m)$ on formal power series in $x = e^{2\pi i z}$, $\operatorname{Im} z > 0$, by

$$\sum_{n \geqslant 1} a(n)\,x^n \mid U(m) = \sum_{n \geqslant 1} a(mn)\,x^n.$$

Let $a_f(n)$ be the nth Fourier coefficient of a modular form f and $a_f(n) = 0$ if n is not an integer. By $d \mid N$, we mean that d is a positive divisor of N.

For $p \mid M$, let

$$W(p) = \left(\begin{pmatrix} pa & b \\ 4Mc & p \end{pmatrix}, p^{-1/4}(4Mcz + p)^{1/2} \right),$$

where a, b, c are integers with $b \equiv 1 \pmod p$ and $p^2 a - 4Mcb = p$ and

$$W(4) = \left(\begin{pmatrix} 4a & b \\ 4Mc & 4 \end{pmatrix}, 2^{1/2}\varepsilon^{-1/2}(-1)^{[(k+1)/2]}\,(Mcz+1)^{1/2} \right),$$

where a, b, c are integers with $b \equiv 1 \pmod 4$ and $16a - 4Mcb = 4$ be the W-operators in $S_{k+1/2}(N, \chi)$.

For $p \mid N$, let $W_p = \left(\begin{smallmatrix} pa & b \\ Nc & p \end{smallmatrix}\right)$, where a, b, c are integers satisfying $p^2 a - Ncb = p$, be the W-operator on $S_{2k}(N)$. For $p \nmid N$, let $T(p^2)$ and $T(p)$

be the Hecke operators on $S_{k+1/2}(N, \chi)$ and $S_{2k}(N)$, respectively. We have

$$f \mid T(p^2) = \sum_{n \geqslant 1} \left(a_f(p^2 n) + \chi(p)\left(\frac{\varepsilon(-1)^k n}{p}\right) p^{k-1} a_f(n) + p^{2k-1} a_f(n/p^2)\right) x^n$$

and

$$F \mid T(p) = \sum_{n \geqslant 1} (a_F(pn) + p^{2k-1} a_F(n/p))\, x^n,$$

where $f \in S_{k+1/2}(N, \chi)$ and $F \in S_{2k}(N)$. The $C(p)$-operator on $S_{2k}(N)$ is given by

$$C(p) = U(p) + W_p U(p) W_p + p^{k-1} W_p, \qquad p \mid N.$$

Let $S^{\text{new}}_{k+1/2}(N, \chi)$ and $S^{\text{old}}_{k+1/2}(N, \chi)$ be the spaces of newforms and oldforms of $S_{k+1/2}(N, \chi)$, respectively. We write $S_{k+1/2}(N, 1) = S_{k+1/2}(N)$.

For details see [1–3].

3. The $C(p^2)$-Operator on $S_{k+1/2}(N, \chi)$

Let

$$w_{p,\chi} = \begin{cases} U(t)^{-1} w_p U(t) & \text{if } p \mid M \\ W(4) & \text{if } p = 2, \end{cases}$$

where $w_p = p^{-k/2+1/4} U(p)\, W(p)$. The operator $w_{p,\chi}$, $p \mid N$ is a hermitean involution on $S_{k+1/2}(N, \chi)$ (cf. [3, Lemma 3]).

Now we define the analogous Pizer operator on $S_{k+1/2}(N, \chi)$ as follows.

$$C(p^2) = \begin{cases} U(p^2) + w_{p,\chi} U(p^2)\, w_{p,\chi} + p^{k-1} w_{p,\chi} & \text{if } p \mid M \\ \left.\begin{matrix} p^{1-k} U(p^4)\, w_{p,\chi} + p^{k-1} w_{p,\chi} \text{ on } S^{\text{new}}_{k+1/2}(d, \chi) \mid U(p^2) \\ 3 w_{p,\chi} U(p^2)\, w_{p,\chi} - U(p^2) + p^{k-1} w_{p,\chi} \text{ otherwise} \end{matrix}\right\} & \text{if } p = 2, \end{cases}$$

where $d \mid M$.

Then, it is clear that the operator $C(p^2)$, $p \mid N$ is an endomorphism of $S_{k+1/2}(N, \chi)$. Since $\mathscr{S}_{k,N,\chi} = U(t)^{-1} \mathscr{S}_{k,N}$ (cf. [3, Theorem 7]) using (3.9) of [3], we have

$$w_{p,\chi} \mathscr{S}_{k,N,\chi} = \mathscr{S}_{k,N,\chi} W_p (p \mid M). \tag{1}$$

Also $U(p^2)\, \mathscr{S}_{k,N,\chi} = \mathscr{S}_{k,N} U(p)$. Therefore,

$$C(p^2)\, \mathscr{S}_{k,N,\chi} = \mathscr{S}_{k,N,\chi} C(p), \qquad p \mid M \text{ on } S_{k+1/2}(N, \chi). \tag{2}$$

Note that, if N is even and $p=2$, (1) holds only on $S^{\text{new}}_{k+1/2}(N, \chi)$ and hence we do not have a single definition for $C(4)$ on $S_{k+1/2}(N, \chi)$. In the next section we will obtain relation (2) for $p=2$, if N is even.

4. Diagonalising the Space $S_{k+1/2}(N, \chi)$

First of all we shall prove a result which is completely analogous to the corresponding result in the case of integral weight, namely,

Lemma 1. *$C(p^2)=U(p^2)$, $p \mid N$ on $S^{\text{new}}_{k+1/2}(N, \chi)$.*

Proof. On $S^{\text{new}}_{k+1/2}(N, \chi)$,

$$U(p^2) = -p^{k-1} w_{p,\chi}, \qquad p \mid N \text{ (cf. [3, Theorem 1])}.$$

Since, $w_{p,\chi}$, $p \mid N$ preserves $S^{\text{new}}_{k+1/2}(N, \chi)$, we have $w_{p,\chi} U(p^2)\, w_{p,\chi} = U(p^2)$, $p \mid N$ on $S^{\text{new}}_{k+1/2}(N, \chi)$, which proves the lemma.

Remark 1. The above lemma says that the operator $C(p^2)$, $p \mid N$ is a new operator only on $S^{\text{old}}_{k+1/2}(N, \chi)$.

Lemma 2. *If N is even,*

$$C(4)\, \mathscr{S}_{k,N,\chi} = \mathscr{S}_{k,N,\chi}\, C(2) \qquad \textit{on} \quad S^{\text{old}}_{k+1/2}(N, \chi).$$

Proof. We know that the space $S^{\text{old}}_{k+1/2}(N, \chi)$ can be decomposed as a direct sum of common eigenspaces $V_{k+1/2}$ generated by forms $f \mid U(d^2)$, $d \mid N/N'$, where $f \in S^{\text{new}}_{k+1/2}(N', \chi)$ $(N' \mid N, N' \neq N)$ such that $F = f \mid \mathscr{S}_{k,N,\chi} \in S^{\text{new}}_{2k}(N')$ is a normalized newform and the forms $F \mid U(d), d \mid N/N'$ generate the eigenspace $V_{2k} = V_{k+1/2} \mid \mathscr{S}_{k,N,\chi}$ in $S^{\text{old}}_{2k}(N)$ (cf. [3]).

Hence it suffices to prove the commutative rule on $V_{k+1/2}$. If N' is even, by Lemma 1, we have $C(4) = U(4)$ on $S^{\text{new}}_{k+1/2}(N', \chi)$ and hence, $C(4)\, \mathscr{S}_{k,N,\chi} = \mathscr{S}_{k,N,\chi}\, C(2)$, since,

$$C(2) = U(2) \qquad \text{on } S^{\text{new}}_{2k}(N').$$

So, we assume that N' is odd. Since

$$f \mid W(4)\, U(4)\, W(4) = f \mid (T(4) - 2^{-1} f \mid U(4))$$

and

$$f \mid U(4)\, W(4) = 2^k f \qquad \text{(cf. [3, Lemma 4])}$$

$$\begin{aligned} f \mid C(4) &= 3(f \mid T(4) - 2^{-1} f \mid U(4)) - f \mid U(4) + 2^{-1} f \mid U(4) \\ &= 3f \mid T(4) - 2f \mid U(4). \end{aligned}$$

Therefore,

$$\begin{aligned}
&f \mid C(4) \mid \mathscr{S}_{k,N,\chi} \\
&\quad = 3F \mid T(2) - 2F \mid U(2) \\
&\quad = 3\left(F \mid U(2) + 2^{k-1}F \left| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right.\right) - 2F \mid U(2) \\
&\quad = F \mid U(2) + 3 \cdot 2^{k-1}F \left| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right. \\
&\quad = F \mid C(2) \\
&\quad = f \mid \mathscr{S}_{k,N,\chi} \mid C(2).
\end{aligned}$$

Also, by definition, on $S^{\text{new}}_{k+1/2}(N', \chi)$,

$$\begin{aligned}
U(4)\, C(4) &= U(4)(2^{1-k}U(16)\, W(4) + 2^{k-1}W(4)) \\
&= 2T(4)^2 - 2^{2k-1} - T(4)\, U(4).
\end{aligned}$$

Thus

$$\begin{aligned}
&f \mid U(4)\, C(4) \mid \mathscr{S}_{k,N,\chi} \\
&\quad = 2F \mid T(2)^2 - 2^{2k-1}F - F \mid T(2)\, U(2) \\
&\quad = 2F \mid (U(2)^2 - 2^{2k-1} - 2^{k-1}U(2)\, W(2) + 2^{2k-2}) \\
&\qquad - 2^{2k-1}F - F \mid (U(2)^2 - 2^{2k-1}) \\
&\quad = F \mid U(2)^2 - 2^{2k-1}F - 2^kF \mid U(2)\, W(2) \\
&\quad = F \mid U(2)\, C(2) \\
&\quad = f \mid \mathscr{S}_{k,N,\chi} \mid U(2)\, C(2)
\end{aligned}$$

and hence

$$\begin{aligned}
&f \mid C(4) \mid \mathscr{S}_{k,N,\chi} \\
&\quad = f \mid \mathscr{S}_{k,N,\chi} \mid C(2) \qquad \text{for all} \quad f \in V_{k+1/2},
\end{aligned}$$

since

$$U(p^2)\, C(4) = C(4)\, U(p^2) \qquad \text{on} \quad S_{k+1/2}(N, \chi)\ (p \neq 2).$$

This completes the proof of Lemma 2.

Using (2) and Lemma 2, we have the following

THEOREM 1. *For $p \mid N$,*

$$C(p^2)\,\mathscr{S}_{k,N,\chi} = \mathscr{S}_{k,N,\chi}\,C(p) \qquad \text{on } S_{k+1/2}(N,\chi).$$

THEOREM 2. *The space $S_{k+1/2}(N,\chi)$ can be decomposed into a direct sum of common eigenspaces of all Hecke operators $T(p^2)(p \nmid N)$ and $C(p^2)(p \mid N)$, each of dimension one. If f is a common eigenform with respect to all $T(p^2)$ and all $C(p^2)$, say with eigenvalues a_p respectively, then any $F \in S_{2k}(N)$, which corresponds to f via the Shimura liftings, satisfies $a_F(p) = a_p a_F(1)$; in particular, any such F can be normalized.*

Proof. Since $\mathscr{S}_{k,N,\chi}$ is an isomorphism the proof follows from Theorem 1 and Theorem 3.10 of [4].

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